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Stabilization of dark solitons in the cubic Ginzburg-Landau equation

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The existence and stability of exact continuous-wave and dark-soliton solutions to a system consisting of the cubic complex Ginzburg-Landau (CGL) equation linearly coupled with a linear dissipative equation is studied. We demonstrate the existence of vast regions in the system's parameter space associated with stable dark-soliton solutions, having the form of the Nozaki-Bekki envelope holes, in contrast to the case of the conventional CGL equation, where they are unstable. In the case when the dark soliton is unstable, two different types of instability are identified. The proposed stabilized model may be realized in terms of a dual-core nonlinear optical fiber, with one core active and one passive.

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I. INTRODUCTION

It is commonly known that the complex cubic Ginzburg-Landau (CGL) equation plays an important role in pattern-formation theory [1]. Various exact solitary-wave solutions to this equation are available [2], two of which have the form of stationary localized pulses: bright solitons (where we realize the word ''soliton'' in a loose sense, without implying exact integrability) [3] and dark ones (also known as ''envelope holes''), which exist against a continuous-wave (cw) background [4].

Because of the presence of linear gain in the CGL equation, the bright solitons of the CGL equation are unstable, as their background is unstable. However, in the case of bright pulses, a stabilization scheme was proposed [5] and then checked by direct numerical simulations [6]. The scheme is based on linearly coupling the CGL equation to an extra linear dissipative equation, and may find direct physical applications in the context of nonlinear fiber optics, describing a dual-core optical fiber, in which an active nonlinear core carries the gain, a parallel-coupled passive linear core being lossy. It has very recently been shown [7] that this stabilized model provides for stable transmission of bright pulses in long-distance optical links with *normal* dispersion.

Apart from the bright solitons, the dark solitons of the CGL equation have also been a subject of interest: both numerical simulations [8,9] and analytical works [2,10] revealed the existence and importance of various hole solutions. In particular, the dark solitary waves discovered by Nozaki and Bekki (NB) [4] have attracted attention, although they are known to be generally unstable [11] (actually there exists a very narrow parametric region where they are stable [12]). This is due to the fact that they play a significant dynamical role in a large part of the parameter space. In fact, they are also important in the case when they are unstable, because in that case they may control the onset of spatiotemporal chaos (turbulence) in the system [13]. Patterns similar to the NB envelope holes have been identified in various experiments with traveling-wave convection and coupled wakes [14]. Moreover, direct experimental evidence of the existence of the NB dark solitary waves proper has recently been reported in an experimental study of hydrothermal waves in a laterally heated fluid layer [15], as well as in wakes [16] and in a chemical system [17].

Generally, the search for a physically realistic system that can support stable solitary hole (dark) solitons is of considerable interest, both for physical applications, and also in the context of controlling spatiotemporal chaos [18]. In particular, the dynamics of dark solitons and the underlying cw background can be studied in various perturbed versions of the nonlinear Schrödinger (NLS) equation, which resemble the CGL equation (see, e.g., [19] and references therein). For example, in a NLS equation, which incorporates linear gain and nonlinear absorption as small perturbations, the cw background can be stabilized, but the dark soliton is still unstable [20], in agreement with the instability of the NB dark soliton in the CGL equation. In this case, a stabilization technique for the dark soliton, based on the inclusion of nonlinear gain, has been proposed [21]. However, these approaches have been developed only for slightly perturbed *unchirped* dark solitons of the NLS equation, rather than for the exact chirped NB dark solitary waves of the CGL equation. A more general approach to the stabilization of the dark soliton, similar to the introduction of nonlinear gain, is to add quintic terms as a small perturbation [13]. In this connection, it is relevant to mention that the pure CGL equation is rather degenerate: the quiescent hole (dark-soliton) solution is a member of a continuous solution family of moving gray solitons (those with a dip that does not reach zero). The analysis developed in Ref. [13] has demonstrated that this family is *structurally unstable*, disappearing after the addition of a small quintic term, while a quiescent dark soliton may survive and become stable.

In this paper, we aim to develop another approach to the stabilization of NB dark solitons, based on the above-mentioned scheme that was successfully used to stabilize bright solitons of the CGL equation [5–7]. To this end, we introduce a system of two linearly coupled equations, in

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which the first one is the CGL equation, while the second is a simple linear dissipative equation. We display conditions for the existence of explicitly found exact cw and NB dark-soliton solutions. We also study their stability, concluding that they are stable in vast regions of the parameter space. In particular, as concerns the dark solitons, we demonstrate by means of direct simulations that infinitely long propagation distances may be achieved. Thus, we present a stabilization scheme for nonperturbative dark solitary waves of the CGL equation.

The paper is organized as follows. In Sec. II we formulate the model and present its exact cw and dark-soliton solutions. In Secs. III and IV we study the stability of the cw and dark solitons, respectively. The results obtained are summarized in Sec. V.

II. THE MODEL AND ITS EXACT CONTINUOUS-WAVE AND DARK-SOLITARY-WAVE SOLUTIONS

We consider the following system of two linearly coupled equations, in which the first one is a CGL equation, while the second component is a simple linear dissipative equation:

$$iu_z+(\tfrac{1}{2}D-i)u_{tt}+|u|^2u-iu=Kv, \tag{1}$$

$$iv_z+i\Gamma v+k_0 v=Ku. \tag{2}$$

The model is formulated in terms of nonlinear fiber optics, where it applies to a dual-core nonlinear fiber coupler [5–7]. Accordingly, u and v are the amplitudes of the electromagnetic waves in the active and passive cores, the evolution variable z is the propagation distance, and t is the so-called retarded time. The CGL Eq. (1) is normalized so that the linear gain and dispersive loss coefficients are equal to 1 i.e., the corresponding effects *cannot* be considered as small perturbations. The parameter D is the group-velocity dispersion coefficient ($D>0$ in the anomalous-dispersion region, and $D<0$ in the normal-dispersion one), K is the coupling constant, Γ is the loss coefficient in the linear core, and k_0 is a phase-velocity mismatch between the two cores.

The simplest nontrivial solution to Eqs. (1) and (2) is a one-parameter continuous-wave solution of the form

$$\begin{aligned} u^{(0)}&=u_0\exp[i(k^{(0)}z-\omega^{(0)}t)],\\ v^{(0)}&=u_0K(-k^{(0)}+k_0+i\Gamma)^{-1}\exp[i(k^{(0)}z-\omega^{(0)}t)], \end{aligned} \tag{3}$$

where the amplitude u_0, the wave number $k^{(0)}$, and the frequency $\omega^{(0)}$ (assumed to be real) are connected through the equation

$$-k^{(0)}-(\tfrac{1}{2}D-i)\omega^{(0)2}+u_0^2-i-(-k^{(0)}+k_0+i\Gamma)^{-1}K^2=0. \tag{4}$$

Using the real and imaginary parts of Eq. (4), we may obtain the following inequalities:

$$1\leqslant\Gamma\leqslant K^2,\quad K^2\geqslant\Gamma^{-1}[\Gamma^2+(-k^{(0)}+k_0)^2], \tag{5}$$

which follow from $(-k^{(0)}+k_0)^2\geqslant0$. Notice that the equalities $\Gamma=1$ and $\Gamma=K^2$, related to Eqs. (5), define an area in the K-Γ plane within which the existence of the cw solution is guaranteed.

Additionally, there exists an exact analytical solution in the form of a stationary envelope hole, which resembles a chirped dark (black) soliton and follows the pattern of the original envelope hole solution of the GL equation [4]:

$$u=u_0\frac{1-\exp(2\eta t)}{[1+\exp(2\eta t)]^{1+i\mu}}\exp[i(kz-\omega t)], \tag{6}$$

$$\begin{aligned} v=&u_0K(-k+k_0+i\Gamma)^{-1}\frac{1-\exp(2\eta t)}{[1+\exp(2\eta t)]^{1+i\mu}}\\ &\times\exp[i(kz-\omega t)]. \end{aligned} \tag{7}$$

Notice that the asymptotic amplitude (at $|t|\to\infty$) of the dark-soliton solution coincides with that of the cw solution, and thus the dark soliton may indeed be regarded as a localized dip in the cw background. The ''chirp'' μ in the exact solution (6), (7) is

$$\mu=-\tfrac{3}{4}D-\sqrt{\tfrac{9}{16}D^2+2}, \tag{8}$$

while the remaining parameters, namely, the amplitude u_0, the inverse width η of the dark pulse, and the wave number k, are determined as follows. First, k is to be found from the cubic equation,

$$\begin{aligned} &\mu^2k^3+(D+3\mu-2k_0\mu^2)k^2+[\mu^2(\Gamma^2-K^2+k_0^2)\\ &\quad-2k_0(D+3\mu)]k+(D+3\mu)[\Gamma(\Gamma-K^2)+k_0^2]\\ &\quad+k_0\mu^2K^2=0. \end{aligned} \tag{9}$$

Then, η is expressed in terms of k:

$$\eta^2=\frac{k(\Gamma-1)+k_0}{-\Gamma(D+3\mu)+\mu^2(-k+k_0)}. \tag{10}$$

Finally, the amplitude u_0 and the frequency ω can be found:

$$u_0^2=-3\mu\eta^2(1+\tfrac{1}{4}D^2), \tag{11}$$

$$\omega=-\mu\eta. \tag{12}$$

It is important to mention that the existence condition of the dark-soliton solution is that the roots of Eq. (9) satisfy the condition $\eta^2>0$.

III. STABILITY OF THE CONTINUOUS-WAVE SOLUTION

Stability of the cw solution (3), apart from being interesting by itself, is crucial for the stability of the dark soliton, because the latter cannot be stable unless its cw background is stable. In order to investigate the stability, we consider solutions to Eqs. (1) and (2) of the form $u=u^{(0)}+u^{(1)}$, $v=v^{(0)}+v^{(1)}$, where $u^{(1)}$ and $v^{(1)}$ are small perturbations. Then, we linearize the equations in $u^{(1)}$ and $v^{(1)}$, substitute $u^{(1)}=u_1\exp[i(k^{(0)}z-\omega^{(0)}t)]$, $v^{(1)}=v_1\exp[i(k^{(0)}z-\omega^{(0)}t)]$, and split the unknown functions u_1 and v_1 into their real and imaginary parts, $u_1=u_1^{(r)}+iu_1^{(i)}$ and $v_1=v_1^{(r)}+iv_1^{(i)}$. Finally, we look for solutions to the resulting equations in the form $u_1^{(r)},u_1^{(i)},v_1^{(r)},v_1^{(i)}\propto\exp[i(Qz-\Omega t)]$, where Q and Ω are the wave number and frequency of the perturbation. Thus we

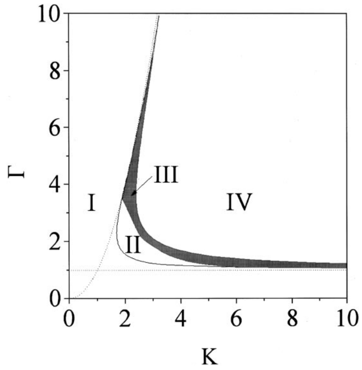

FIG. 1. Regions of existence and stability of the cw solution and stability region of the exact dark-soliton solution in the Γ-K parameter plane. The existence region of the cw, solution is confined by the dashed lines $\Gamma=1$ and $\Gamma=K^2$ [see Eq. (5)]. In the regions I and IV, the cw solution is unstable. In the region II, the cw solution is stable, while the dark-soliton one is not. In the region III, both the cw and dark-soliton solutions are stable.

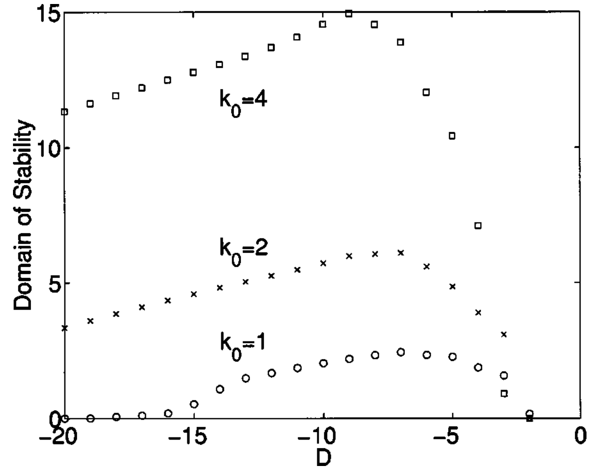

FIG. 2. The domain of stability of the cw solution (corresponding to the union of regions II and III of Fig. 1) as a function of D, for $k_0=1$, 2, and 4. The area turns out to be an almost linear function of k_0 (at $k_0=0$ the stability area does not exist), and, as a function of D, it has a maximum at $D\approx-7$, which is moving slightly deeper into the normal-dispersion region for larger values of k_0.

arrive at the dispersion relation,

$$Q^4+2i(\gamma-\Gamma)Q^3+(-\alpha\beta-\gamma^2-\delta^2-\Gamma^2+4\gamma\Gamma-2K^2)Q^2$$
$$+2i[\Gamma(\alpha\beta+\gamma^2)-\gamma(\delta^2+\Gamma^2)+K^2(\Gamma-\gamma)]Q$$
$$+(\alpha\beta+\gamma^2)(\Gamma^2+\delta^2)+K^2[K^2-\delta(\alpha+\beta)-2\gamma\Gamma]=0, \tag{13}$$

where

$$\alpha\equiv-k-\tfrac{1}{2}D(\omega^{(0)2}+\Omega^2)+3u_0^2+2i\omega^{(0)}\Omega, \tag{14}$$

$$\beta\equiv\alpha-2u_0, \tag{15}$$

$$\gamma\equiv1-\omega^{(0)2}-\Omega^2-iD\omega^{(0)}\Omega, \tag{16}$$

$$\delta\equiv-k^{(0)}+k_0. \tag{17}$$

Equation (13) is a quartic equation with complex coefficients, which, in general, has four complex roots Q_j ($j=1,2,3,4$). The stability region, which is defined by the condition $\mathrm{Im}\{Q_j\}>0$ for $-\infty<\Omega<+\infty$, can be determined numerically by using the following procedure. As is seen from Eqs. (14)–(17), the coefficients of the quartic equation depend on the cw parameters u_0, $k^{(0)}$, and $\omega^{(0)}$, one of which is free [recall that the cw solution (3) has one free parameter]. This free parameter, however, can be determined by requiring that the dark soliton's asymptotic amplitude coincide with that of the cw background. Thus, we may find the roots of the cubic equation (9) that satisfy the condition $\eta^2>0$ [see Eq. (10)] to determine the wave number $k^{(0)}$ and then utilize Eqs. (10)–(12) to find the amplitude u_0 and frequency $\omega^{(0)}$. In way, the dependence of the coefficients of the quartic equation (13) on the parameters of the cw background is known. In addition, setting $D=-7$ and $k_0=2$ (we will explain this choice below), the coefficients of the quartic equation (13) are all known, and thus we may find its roots numerically, to determine the stability region.

In Fig. 1, the stability region thus found is displayed in the Γ-K parametric plane: it is a union of two regions, II and III in Fig. 1, having a ''boomerang'' shape. Notice that the left and bottom borders of the stability region tend asymptotically to the above-mentioned curves $\Gamma=K^2$ and $\Gamma=1$, respectively, which are the existence boundaries for the cw solution [see Eq. (5)]. In the other regions in Fig. 1, i.e., I and IV, the condition $\mathrm{Im}\{Q_j\}>0$ for $-\infty<\Omega<+\infty$ is not fulfilled and thus the cw background is unstable there.

It is also important how the parameters D and k_0 affect the stability of the cw solution. To this end, Fig. 2 shows the area of the stability region from Fig. 1 as a function of D, for several values of k_0 ($k_0=1$, 2, and 4). We stress that the cw is unstable if $k_0=0$, and thus nonzero values of k_0 are required to provide for the stability of the cw solution. On the other hand, it is seen from Fig. 2 that the area of the stability region depends almost linearly on k_0, and, as a function of D, it has a maximum in the interval $-9<D<-7$ (inside the *normal*-dispersion region). Also, it is important to notice that if $D>D_{\mathrm{cr}}\approx-2.5$ the stability region of the cw solution ceases to exist and thus there is no chance to have a stable cw (or dark-soliton) solution close to the zero-dispersion point ($D=0$) and in the anomalous-dispersion regime ($D>0$).

According to these results, the choice of the aforementioned values $D=-7$ and $k_0=2$ for the numerical simulations to be displayed below is quite natural: a nonzero value of the phase-velocity mismatch is needed to ensure the existence of the stability region, and $D=-7$ provides for a large size of the stability domain. Note that it is reasonable to choose a relatively small value of k_0, having regard to the feasibility of the implementation of the stabilization scheme in a real experiment (using, e.g., delay lines with periodically placed short second-core segments that are parallel-coupled to the main core [7]).

IV. STABILITY OF THE DARK SOLITONS

As mentioned in Sec. II, a necessary condition for the existence of the dark soliton is $\eta^2>0$ [see Eq. (10)], for

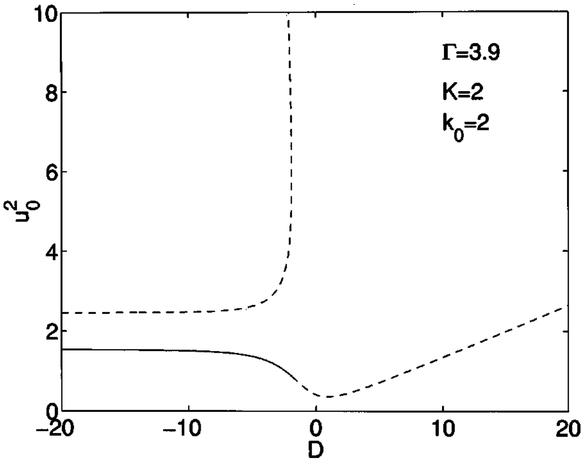

FIG. 3. The amplitude u_0^2 of the dark-soliton solution, corresponding to the three roots of the cubic equation (9), as a function of the parameter D, for $\Gamma=3.9$, $K=2$, and $k_0=2$. The solid line (corresponding to a smaller amplitude) shows the stable dark soliton, while the dashed lines show unstable ones. Notice that stable dark pulses exist, provided that $D<D_{cr}\approx-2.5$.

every real root of the cubic equation (9). Apparently, Eq. (9) may have one or three real roots, leading to one or three different dark solitons. They are characterized by different amplitudes u_0, which, in general, depend on the values of the parameters of the GL equations (1) and (2). A typical example is shown in the bifurcation diagram of Fig. 3, where the amplitudes u_0^2 of the dark solitons are plotted vs the dispersion parameter D (the other parameters are $\Gamma=3.9$, $K=2$, $k_0=2$, corresponding to region III in Fig. 1, where the cw solution is stable). As is seen, at $D>-2.5$ ($D<-2.5$) the cubic equation has one (three) real roots, leading to a single (three) soliton(s), whose stability is the issue.

Direct numerical simulations demonstrate that the dark soliton corresponding to the solid curve in Fig. 3 is stable, while the ones corresponding to the dashed curves are unstable. Notice that, according to this result, the stable dark soliton is that with the *smallest* possible amplitude u_0.

In the simulations, the stability is realized as robust propagation of the soliton over infinitely long distance. This is what we have observed at every tested point (K,Γ) in the shaded region (region III) in Fig. 1. This is illustrated by Fig. 4(a), showing a typical example of a stable dark soliton: the pulse propagates undistorted up to $z=2000$ in the case $K=3$, $\Gamma=2$ (the other parameters are $D=-7$ and $k_0=2$, as fixed above).

If the coupling parameter K is decreased, so as to enter region II in Fig. 1, the bifurcation diagram shown in Fig. 3 is only slightly changed: the curves maintain their shape but they are both displaced, increasing the separation between them. Nevertheless, the behavior of the dark pulses is drastically changed: although the more stable one is again the dark soliton corresponding to the smaller amplitude u_0, the propagation distance is not infinitely long in this case. This is demonstrated in Fig. 4(b), where, as an example, the evolution of the dark pulse is shown for $K=2$ and $\Gamma=2$ (the other parameters are again $D=-7$ and $k_0=2$). As is seen, at the first stage, the dark soliton propagates undistorted, but then it collapses at a propagation distance $z=200$, in contrast to the previous case (inside region III in Fig. 1), where the propagation distance was infinitely long. Notice that no change in the shape of the underlying cw background is observed, in compliance with the fact that the cw solution remains stable in this region.

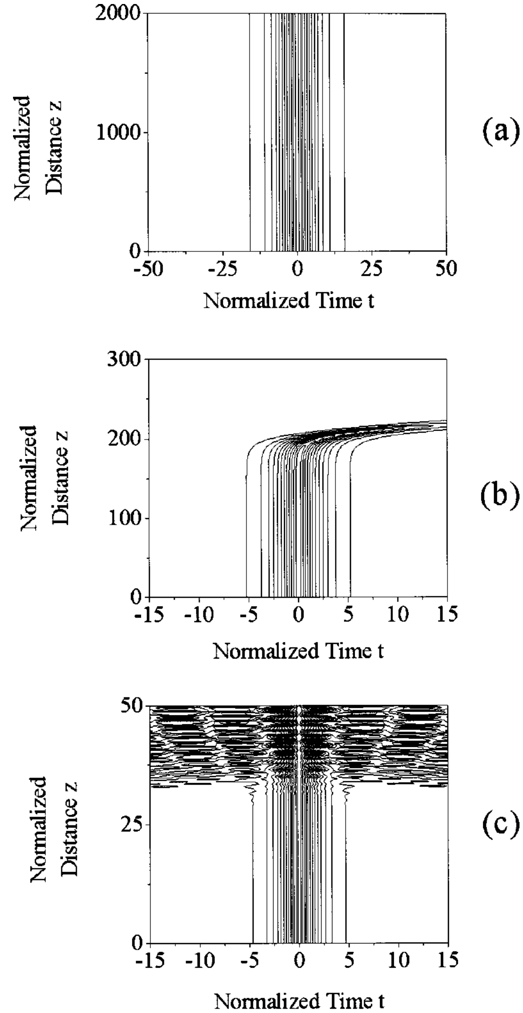

FIG. 4. Contour plots of the evolution of dark solitary waves for $D=-7$ and $k_0=2$. (a) A stable dark solitary wave for $K=3$, $\Gamma=2$ (region III in Fig. 1), which propagates undistorted throughout the total simulated distance $z=2000$. (b) A dark solitary wave for $K=2$, $\Gamma=2$ (region II in Fig. 1), which propagates undistorted up to the distance $z=200$ and then collapses. (c) A dark solitary wave for $K=1.6$, $\Gamma=2$ (region I in Fig. 1) on top of an unstable cw pedestal. A quick onset of a turbulent state is observed.

Finally, we consider the evolution of the dark solitary waves for values of Γ and K belonging to region I or IV in Fig. 1. As shown in Sec. III, the cw solution is unstable in these regions and, as a result, in this case the dark soliton is expected to be subject to a background instability. This prediction is confirmed by Fig. 4(c), where the evolution of a dark soliton is shown for $K=1.6$, $\Gamma=2$ (which corresponds to region I), $D=-7$, and $k_0=2$. As is seen, the dark pulse and the underlying cw background experience a ''laminar'' evolution initially. Nevertheless, after a very short propagation distance ($z\approx30$) the cw becomes modulationally unstable and, as a result, the ''laminar'' propagation ends by a transition to an obviously turbulent state. Similar results were obtained for all the values of Γ and K belonging to region IV in Fig. 1.

V. CONCLUSIONS

In conclusion, we have found exact analytical continuous-wave and dark-soliton solutions in a system of two linearly coupled equations, the CGL equation and a linear dissipative one. The dark solitons have the form of the Nozaki-Bekki envelope holes of the conventional CGL equation. We have found conditions for existence of the cw and dark-soliton solutions, and we have studied their stability in detail by means of numerical simulations. In contrast to the case of the single-component CGL equation, we have found vast regions in the parameter space where stable cw states and dark solitons exist. In these regions, the dark solitons propagate over infinitely long distances without any distortions in their shape. Additionally, we have found that, outside their stability region, but inside the stability region of the underlying cw background, the dark solitons collapse. Outside the stability region of the cw pedestal, the dark solitons are subject to modulation instability.

These results clearly demonstrate that the nonperturbative dark solitary waves of the CGL equation can be stabilized. The proposed stabilization scheme can be directly implemented in a dual-core nonlinear optical fiber, with one active nonlinear core (where the evolution is described by the CGL equation) and one passive linear core (described by the linear dissipative equation).

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